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DECOMPOSITION OF INJECTIVE MODULES RELATIVE TO A TORSION THEORY

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ABSTRACT

If R is a right noetherian ring, the decomposition of an injective module, as a direct sum of uniform submodules, is well known. Also, this property characterises this kind of ring. M. L. Teply obtains this result for torsion-free injective modules. The decomposition of injective modules relative to a torsion theory has been studied by S. Mohamed, S. Singh, K. Masaike and T. Horigone. In this paper our aim is to determine those rings satisfying that every torsion-free τ-injective module is a direct sum of τ-uniform τ-injective submodules and also to determine those rings with the same property for every τ-injective module.

1. Preliminaries

All rings considered have non-zero identity elements, are associative, but not necessarily commutative. All modules are unitary and are right modules over the ring.

Let R be a ring. Let Mod-R denote the category of right R-modules. The pair $\tau = (\mathcal{T}, \mathcal{F})$ will always denote a hereditary torsion theory on Mod-R and $\mathcal{L}$ will be its associated Gabriel filter. Denote by $t(M)$ the largest τ-torsion submodule of a module M. t is the associated torsion radical of the torsion theory τ.

If M is a module and $N \leqq M$ a submodule, then

$$\mathrm{Cl}_\tau^M(N) = \{x \in M \mid (N : x) \in \mathcal{L}\}$$

will be the τ-closure of N in M.

A submodule N of M is τ-dense (resp. τ-closed) in M if $\mathrm{Cl}_\tau^M(N) = M$ (resp. $\mathrm{Cl}_\tau^M(N) = N$).

Let $\mathcal{C}_\tau(M)$ denote the complete modular lattice of all τ-closed submodules of M. The following lemma is well known.

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LEMMA 1.1. *Let M be a module and $N \leqq M$ a submodule. If $\mathscr{C}_\tau(M)$ is a complemented lattice, then $\mathscr{C}_\tau(N)$ and $\mathscr{C}_\tau(M/N)$ are complemented.*

A module M is τ-simple if $\mathscr{C}_\tau(M)=\{t(M),M\}$ and it is not τ-torsion. It is called τ-critical if it is τ-simple and τ-torsion-free.

A submodule N of M is τ-essential in M if N is essential in M and M/N is τ-torsion.

A non-zero module M is τ-uniform if for every non-zero submodule N, N is τ-essential in M.

LEMMA 1.2. *A module M is τ-uniform if and only if M is either τ-critical or τ-torsion and uniform.*

A module M is τ-injective if for every right ideal I of the filter $\mathscr{L}$, the natural group homomorphism $\mathrm{Hom}_R(R,M)\to \mathrm{Hom}_R(I,M)$ is surjective. M is τ-injective if and only if M has not proper τ-essential extensions. For every module M there is a τ-essential extension $E_\tau(M)$ of M which is τ-injective, the τ-injective hull of M. This is unique up to isomorphism.

Denote by

$$T_\tau : \text{Mod-}R \to (\text{Mod-}R)/\mathscr{T}$$

the canonical quotient functor, and by

$$S_\tau : (\text{Mod-}R)/\mathscr{T} \to \text{Mod-}R$$

its right adjoint functor. As is well known T_τ is exact, and of course S_τ is left exact. Write $Q_\tau = S_\tau T_\tau$ and

$$\psi : 1 \Rightarrow Q_\tau = S_\tau T_\tau$$

for the unit of the adjunction. $Q_\tau(R)$ is a ring and $\psi(R)$ is a ring homomorphism.

Năstăsescu shows, in [9], that $\mathscr{C}_\tau(M)$ satisfies the ascending chain condition if and only if $T_\tau(M)$ is a noetherian object in $(\text{Mod-}R)/\mathscr{T}$.

For additional information on torsion theories, the reader is referred to [3] and [10].

2. τ-Semiartinian modules

Let M be a module. We define the τ-socle, $\mathrm{Soc}_\tau(M)$, of M as the τ-closure of the sum of all τ-simple submodules of M. A module M is called τ-semicritical if $\mathrm{Soc}_\tau(M)=M$ and τ-semiartinian if every non-zero quotient module of M has non-zero τ-socle. The ring R is called τ-semicritical (resp. τ-semiartinian) if the right module R_R is τ-semicritical (resp. τ-semiartinian). R is τ-semiartinian if and only if $\mathrm{Soc}_\tau(M)$ is essential in M for every module M.

LEMMA 2.1. *The following statements are equivalent for every module M:*

(a) *M is τ-semicritical.*

(b) *$\mathscr{C}_\tau(M)$ is a complemented lattice and every τ-closed non-τ-torsion submodule of M contains a τ-simple submodule.*

(c) *For every proper τ-closed submodule N of M there is a τ-simple submodule K of M such that $N \cap K = t(K)$.*

PROOF. It is similar to [1; Corollary 3.11] using: M is τ-simple if and only if $M/t(M)$ is τ-critical.

LEMMA 2.2. *The following conditions are equivalent:*

(a) *R is a τ-semicritical ring.*

(b) *Every module is τ-semicritical.*

(c) *Every τ-torsion-free module is τ-semicritical.*

(d) (Mod-R)/$\mathscr{T}$ *is discrete spectral.*

PROOF. (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c). They are analogous to [1; Theorem 3.12].

(a)$\Leftrightarrow$(d). It is obvious.

PROPOSITION 2.3. *Let τ be a torsion theory. The following statements are equivalent:*

(a) *$\mathscr{L}$ satisfies the ascending chain condition.*

(b) *Every τ-torsion τ-injective module is a direct sum of τ-uniform τ-injective submodules.*

(c) *Every direct sum of τ-torsion τ-injective modules is τ-injective.*

PROOF. See [8; Theorem 7] or [7; Theorem 1].

The following proposition is a refinement to [7; Theorem 2] by Masaike and Horigone. It will be fundamental in our development. But first, we remember that a noetherian torsion theory, [5; page 28], is a torsion theory such that the associated filter has the property: if $I_1 \leqq I_2 \leqq \cdots$ is an ascending chain of right ideals whose union is in $\mathscr{L}$, then I_n is in $\mathscr{L}$ for some n.

PROPOSITION 2.4. *Let τ be a noetherian torsion theory. The following conditions are equivalent:*

(a) *$\mathscr{L}$ satisfies the ascending chain condition and R is τ-semiartinian.*

(b) *Every τ-injective module is an essential extension of a τ-injective submodule which is a direct sum of τ-uniform τ-injective submodules.*

PROOF. (a)$\Rightarrow$(b). Let M be a τ-injective module. Since $t(M)$ is τ-closed in M, then $t(M)$ is τ-injective. Then $t(M)$ is a direct sum of τ-uniform τ-injective submodules by Proposition 2.3. We may assume that $t(M)$ is not an essential submodule of M. Let N be the complement of $t(M)$ in M. Since N has non-essential extensions in M, N is τ-injective. In view of [1; Corollary 3.2], it is possible to obtain a family of τ-uniform τ-injective submodules $\{N_\alpha \mid \alpha \in \Lambda\}$ of N such that $\mathrm{Soc}_\tau(N) = \mathrm{Cl}_\tau^M(\bigoplus\{N_\alpha \mid \alpha \in \Lambda\})$. Since R is τ-semiartinian then $\mathrm{Soc}_\tau(N)$ is essential in N. On the other hand, since τ is a noetherian torsion theory, [3; Proposition 14.4], then $\bigoplus\{N_\alpha \mid \alpha \in \Lambda\}$ is τ-injective and so $(\bigoplus\{N_\alpha \mid \alpha \in \Lambda\}) \bigoplus t(M)$ is the desired submodule of M.

(b)$\Rightarrow$(a). Since every τ-uniform τ-torsion-free τ-injective module is τ-critical, then every τ-injective module has essential τ-socle, so every module has an essential τ-socle and thus R is τ-semiartinian. If M is τ-torsion τ-injective, then there is an essential τ-injective submodule N of M, which is a direct sum of τ-uniform τ-injective submodules; since every submodule of M is τ-dense then $N = M$; and by Proposition 2.3, $\mathscr{L}$ satisfies the ascending chain condition.

3. The main results

LEMMA 3.1. *The following statements are equivalent*:

(a) *$\mathscr{C}_\tau(M)$ is complemented for every τ-torsion-free module M.*

(b) *The essential submodules of every τ-torsion-free module are τ-dense.*

(c) *$\mathscr{C}_\tau(M)$ is the set of all the complement submodules of M for every τ-torsion-free module M.*

(d) *Every τ-torsion-free τ-injective module is injective.*

(e) *$\mathscr{C}_\tau(R)$ is complemented.*

(f) (Mod-R)/$\mathscr{T}$ *is spectral.*

PROOF. (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c). They are well known.

(a)$\Rightarrow$(e). It is obvious.

(e)$\Rightarrow$(d). Every module M is isomorphic to a quotient of a free module $R^{(\Gamma)}$ for some index set Γ. Then, by Lemma 1.1, it is sufficient to prove that $\mathscr{C}_\tau(R^{(\Gamma)})$ is complemented for any index set Γ. If M is a τ-torsion-free module such that $\mathscr{C}_\tau(M)$ is complemented, then $\mathscr{C}_\tau(M^{(\Gamma)})$ is complemented from [4; Proposition 1.2]. Since $\mathscr{C}_\tau(M)$ is isomorphic to $\mathscr{C}_\tau(M/t(M))$, then this result is immediate for non-necessarily τ-torsion-free modules.

(b) $\Rightarrow$ (d). Let M be a τ-torsion-free τ-injective module and $E(M)$ its injective hull; since M is essential in $E(M)$ and $E(M)$ is τ-torsion-free then $M = E(M)$ is injective.

(d)$\Rightarrow$(a). Since $\mathscr{C}_\tau(M)$ is a lattice isomorphic to $\mathscr{C}_\tau(E_\tau(M))$, we only have to show that $\mathscr{C}_\tau(E_\tau(M))$ is complemented, but this is clear.

(d)$\Rightarrow$(f). It is obvious.

A torsion theory τ is called of finite type if the filter has a cofinal subset of finitely generated right ideals [3; page 141]. If furthermore the functor Q_τ is exact the torsion theory will be called perfect [3; page 156].

THEOREM 3.2. *The following conditions are equivalent for a ring* R:

(a) *Every τ-torsion-free τ-injective module is a direct sum of τ-uniform τ-injective submodules.*

(b) $\mathscr{C}_\tau(R)$ *satisfies the ascending chain condition and* R *is τ-semicritical.*

(c) $\mathscr{C}_\tau(R)$ *satisfies the ascending chain condition and is complemented.*

(d) $\mathscr{C}_\tau(R)$ *satisfies the descending chain condition and is complemented.*

(e) $\mathscr{C}_\tau(R)$ *satisfies the ascending chain condition and every τ-torsion-free τ-injective module is injective.*

(f) $Q_\tau(R)$ *is semisimple artinian.*

(g) τ *is of finite type and* $\mathscr{C}_\tau(R)$ *is complemented.*

(h) τ *is perfect and* $\mathscr{C}_\tau(R)$ *is complemented.*

(i) (Mod-R)/$\mathscr{T}$ *is discrete spectral and* $T_\tau(R)$ *is a noetherian object.*

PROOF. (a)$\Rightarrow$(b). By hypothesis, every τ-torsion-free τ-injective module is τ-semicritical, then, by Lemmas 2.2 and 2.1, R is τ-semicritical and $\mathscr{C}_\tau(R)$ is complemented. Since every τ-torsion-free τ-injective module is injective by Lemma 3.1, so $\mathscr{C}_\tau(R)$ satisfies the ascending chain condition because of [11; Theorem 1.2].

(b)$\Rightarrow$(a). If R is τ-semicritical, then every τ-torsion-free τ-injective module is injective by Lemmas 2.1, 2.2 and 3.1. Let E be a τ-torsion-free τ-injective module, by [11; Theorem 1.2] we have $E=\bigoplus\{E_\alpha \mid \alpha\in\Lambda\}$, where E_α is a uniform injective submodule of E for every $\alpha\in\Lambda$. Since R is τ-semicritical, then E_α is τ-semicritical. There is a τ-critical submodule N of E_α by Lemma 2.1, then since E_α is uniform we have that N is essential in E_α and so N is τ-dense in E_α by Lemma 3.1. As a consequence E_α is τ-uniform τ-injective.

(c)$\Leftrightarrow$(d). See [10; Exercise III.9].

(c)$\Leftrightarrow$(e). It is a consequence of Lemma 3.1.

(c)$\Rightarrow$(b). By Lemmas 2.1, 2.2 and 3.1 we must only prove that if N is a non-zero τ-closed submodule of a τ-torsion-free module M, then N contains a τ-critical submodule. If N is uniform then N is τ-critical, because $\mathscr{C}_\tau(N)$ is complemented by Lemma 1.1. Assume N does not contain uniform submodules, there are two non-zero submodules N_1, N_1' of N such that $N_1\cap N_1'=0$. None of them is uniform, hence N_1' contains again two non-zero submodules N_2, N_2' such that $N_2 \cap N_2' = 0$. In this way we obtain a proper ascending chain of direct sums

$$N_1 < N_1 \oplus N_2 < N_1 \oplus N_2 \oplus N_3 < \cdots.$$

It is clear that

$$\mathrm{Cl}_\tau^M(N_1) < \mathrm{Cl}_\tau^M(N_1 \oplus N_2) < \mathrm{Cl}_\tau^M(N_1 \oplus N_2 \oplus N_3) < \cdots$$

is a proper ascending chain of τ-closed submodules, which is a contradiction.

(b)$\Rightarrow$(c). It is immediate by Lemmas 2.1 and 2.2.

(e)$\Leftrightarrow$(f)$\Leftrightarrow$(g)$\Leftrightarrow$(h). After Lemmas 2.1 and 3.1, they are proved in [12; Theorem 2.1].

(b)$\Leftrightarrow$(i). It is clear.

THEOREM 3.3. *The following statements are equivalent for a ring R:*

(a) *Every τ-injective module is a direct sum of τ-uniform τ-injective submodules.*

(b) *R is right noetherian, τ-semicritical and τ is stable.*

(c) *$Q_\tau(R)$ is semisimple artinian, $\mathscr{L}$ satisfies the ascending chain condition and τ is stable.*

PROOF. (a)$\Rightarrow$(b). Since every injective module is τ-injective, then every injective module is a direct sum of uniform injective submodules and so R is noetherian. On the other hand, since every τ-injective module is a direct sum of τ-uniform τ-injective submodules, then they are τ-semicritical and so R is τ-semicritical. Finally τ is stable as a consequence of [4; Proposition 11.3].

(b)$\Rightarrow$(c). It is obvious by Theorem 3.2.

(c)$\Rightarrow$(a). By Theorem 3.2 $\mathscr{C}_\tau(R)$ satisfies the ascending chain condition, then τ is noetherian [3; Proposition 14.10], and by Proposition 2.4 every τ-injective module E is an essential extension of a τ-injective submodule $\oplus\{E_\alpha \mid \alpha \in \Lambda\}$, where the E_α are τ-uniform τ-injective submodules of E. We define $\Lambda' = \{\alpha \in \Lambda \mid E_\alpha$ is τ-torsion$\}$, then $\alpha \in \Lambda \backslash \Lambda'$ if and only if E_α is τ-torsion-free. It is clear that $K = \oplus\{E_\alpha \mid \alpha \in \Lambda'\}$ is essential in $t(E)$. Since K is τ-injective and τ-dense in $t(E)$, then $K = t(E)$. Since τ is stable, then $H = ((\oplus\{E_\alpha \mid \alpha \in \Lambda \backslash \Lambda'\} \oplus t(E))/t(E)$ is essential in $E/t(E)$. Since $\mathscr{C}_\tau(R)$ is complemented and $E/t(E)$ is τ-torsion-free, then H is a τ-essential submodule of $E/t(E)$ by Lemma 3.1. H is also τ-injective so it is equal to $E/t(E)$. Thus

$$E = \oplus\{E_\alpha \mid \alpha \in \Lambda \backslash \Lambda'\} \oplus t(E) = \oplus\{E_\alpha \mid \alpha \in \Lambda\}.$$

REMARK. The following examples show that (c) does not imply (b) without the condition of stability over τ and that there is a right noetherian and τ-semicritical ring such that τ is not stable.

EXAMPLE 1. Let R be the ring of all 2×2 lower triangular matrices $\left[\begin{smallmatrix} Z & 0 \\ Q & Q \end{smallmatrix}\right]$, where Z is the ring of integers and Q the field of rationals, with the usual addition and multiplication of matrices and let $\mathcal{T}$ be the torsion class consisting of all right modules annihilated by $e_{22}R$, where e_{22} denote the matrix with 1 at the 2,2nd entry and 0 elsewhere. τ is not stable. It is clear that R is τ-semicritical and non-noetherian but $Q_\tau(R)$ is semisimple artinian and $\mathcal{L}$ satisfies the ascending chain condition.

EXAMPLE 2. Let R be the ring of all 2×2 lower triangular matrices $\left[\begin{smallmatrix} F & 0 \\ F & F \end{smallmatrix}\right]$, where F is a field, with the usual addition and multiplication of matrices and let $\mathcal{T}$ be the torsion class consisting of all right modules annihilated by $e_{22}R$. τ is not stable and R is noetherian and τ-semicritical but the τ-injective module

$$M = \begin{bmatrix} 0 & 0 \\ F & F \end{bmatrix}$$

is not a direct sum of τ-uniform τ-injective submodules.

It has been pointed out by the referee that part of Lemma 3.1 is contained in [6].

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